

# BERNSTEIN OPERATORS FOR EXPONENTIAL POLYNOMIALS

J.M. ALDAZ, O. KOUNCHEV AND H. RENDER

**ABSTRACT.** Let  $L$  be a linear differential operator with constant coefficients of order  $n$  and complex eigenvalues  $\lambda_0, \dots, \lambda_n$ . Assume that the set  $U_n$  of all solutions of the equation  $Lf = 0$  is closed under complex conjugation. If the length of the interval  $[a, b]$  is smaller than  $\pi/M_n$ , where  $M_n := \max\{|\operatorname{Im}\lambda_j| : j = 0, \dots, n\}$ , then there exists a basis  $p_{n,k}$ ,  $k = 0, \dots, n$ , of the space  $U_n$  with the property that each  $p_{n,k}$  has a zero of order  $k$  at  $a$  and a zero of order  $n - k$  at  $b$ , and each  $p_{n,k}$  is positive on the open interval  $(a, b)$ . Under the additional assumption that  $\lambda_0$  and  $\lambda_1$  are real and distinct, our first main result states that there exist points  $a = t_0 < t_1 < \dots < t_n = b$  and positive numbers  $\alpha_0, \dots, \alpha_n$ , such that the operator

$$B_n f := \sum_{k=0}^n \alpha_k f(t_k) p_{n,k}(x)$$

satisfies  $B_n e^{\lambda_j x} = e^{\lambda_j x}$ , for  $j = 0, 1$ . The second main result gives a sufficient condition guaranteeing the uniform convergence of  $B_n f$  to  $f$  for each  $f \in C[a, b]$ .

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## 1. INTRODUCTION

Let  $\lambda_0, \dots, \lambda_n$  be complex numbers, let  $\Lambda_n$  be the vector  $(\lambda_0, \dots, \lambda_n)$ , and define the linear differential operator  $L$  with constant coefficients by

$$(1) \quad L = \left( \frac{d}{dx} - \lambda_0 \right) \dots \left( \frac{d}{dx} - \lambda_n \right).$$

Complex-valued solutions  $f$  of the equation  $Lf = 0$  are called *exponential polynomials* or  *$L$ -polynomials*. They provide natural generalizations of classical, trigonometric, and hyperbolic polynomials (see [30]), and the so-called  $\mathcal{D}$ -polynomials considered in [27]. For example, it is well known that one can develop a nice spline theory based on cardinal exponential polynomials (see e.g. [26], [29], [25]) and a satisfactory nonstationary multiresolutional analysis for cardinal exponential splines, see the results in [7], [15], [16], [17]

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and [21], rediscovered in [31]. Another motivation stems from the investigation of a new class of multivariate splines, the so-called polysplines, cf. [15], [18].

Special interest in exponential polynomials has arisen recently within Computer Aided Geometric Design for modelling parametric curves. On the one hand, special systems of exponential polynomials are considered, such as

$$1, x, \dots, x^{n-1}, \cos x, \sin x,$$

(which corresponds to the case  $\lambda_0 = \dots = \lambda_{n-2} = 0$ ,  $\lambda_{n-1} = i$ , and  $\lambda_n = -i$ ), cf. [5], [22], [33] and [6], [4] for further generalizations. On the other hand, a remarkable result is the existence of a so-called *normalized Bernstein basis* in certain classes of extended Chebyshev systems, see [3], [24]. In order to explain this result, let us recall that a subspace  $U_n$  of  $C^n(I)$ , the space of  $n$ -times continuously differentiable complex-valued functions on an interval  $I$ , is called an *extended Chebyshev system for the subset  $A \subset I$*  if  $U_n$  has dimension  $n+1$  and each non-zero  $f \in U_n$  vanishes at most  $n$  times in  $A$  (with multiplicities). A system  $p_{n,k} \in U_n, k = 0, \dots, n$ , is a *Bernstein-like basis* for  $a \neq b \in I$ , if the function  $p_{n,k}$  has a zero of order  $k$  at  $a$ , and a zero of order  $n-k$  at  $b$  for  $k = 0, \dots, n$ . For example, in the polynomial case a Bernstein-like basis  $P_{n,k}$  for  $\{a, b\}$  may be defined explicitly by

$$(2) \quad P_{n,k}(x) := \frac{1}{k!} \frac{1}{(b-a)^{n-k}} (x-a)^k (b-x)^{n-k}.$$

The above-mentioned result in [3], [24] says the following: Assume that the constant function 1 is in  $U_n$ ; clearly then there exist coefficients  $\alpha_k, k = 0, \dots, n$ , such that  $1 = \sum_{k=0}^n \alpha_k p_{n,k}$ , since  $p_{n,k}, k = 0, \dots, n$ , is a basis. The *normalization property* proved in [3] and [24] for a certain class of Chebyshev systems says that the coefficients  $\alpha_k$  are *positive*.

In this paper we shall be concerned with Bernstein-like bases and Bernstein operators for the set of exponential polynomials induced by a linear differential operator  $L$  of the type (1), i.e.

$$(3) \quad U_n = E_{(\lambda_0, \dots, \lambda_n)} := \{f \in C^\infty(\mathbb{R}) : Lf = 0\}.$$

It is easy to see that there exists a Bernstein-like basis  $p_{n,k}, k = 0, \dots, n$  for  $a \neq b$  if and only if  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for the set  $\{a, b\}$ . In order to guarantee that the basis functions  $p_{n,k}, k = 0, \dots, n$ , are *strictly positive* on the open interval  $(a, b)$  it is sufficient to know that  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation and that  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev space for the closed interval  $[a, b]$ . In Section 2 we shall give the following criterion:  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev space for the interval  $[a, b]$  if  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation and  $b-a < \pi/M_n$ , where

$$(4) \quad M_n := \max \{|\operatorname{Im} \lambda_j| : j = 0, \dots, n\}.$$

Having at hand a Bernstein-like basis it is natural to ask whether one can introduce a corresponding Bernstein operator, i.e. an operator of the type

$$(5) \quad B_n f(x) := \sum_{k=0}^n \alpha_k f(t_k) p_{n,k}(x)$$

where the coefficients  $\alpha_0, \dots, \alpha_n$  and the knots  $t_0, \dots, t_n$  have to be defined in a suitable way. Our first main result (Section 3, Theorem 19) states the following: assume that  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation and  $b - a < \pi/M_n$ . If  $\lambda_0 \neq \lambda_1$  are real then there exist unique points  $a = t_0 < t_1 < \dots < t_n = b$ , and unique *positive* coefficients  $\alpha_0, \dots, \alpha_n$ , such that the operator  $B_n : C[a, b] \rightarrow E_{(\lambda_0, \dots, \lambda_n)}$  defined by (5) has the following reproducing property

$$(6) \quad B_n(e^{\lambda_0 x}) = e^{\lambda_0 x} \text{ and } B_n(e^{\lambda_1 x}) = e^{\lambda_1 x}.$$

Of course, the latter property is reminiscent of the well known fact that the classical (polynomial) Bernstein operator  $B_n$  on  $[0, 1]$  satisfies  $B_n 1 = 1$  and  $B_n x = x$ . Note that the assumption  $b - a < \pi/M_n$  is crucial: we give an example of an extended Chebyshev system  $E_{(\lambda_0, \dots, \lambda_n)}$  over an interval  $[a, b]$  (for  $n = 3$ ) such that the condition (6) implies the positivity of the coefficients  $\alpha_0, \alpha_1, \alpha_2, \alpha_3$  but the points  $t_0, t_1, t_2, t_3$  in (5) are not ordered, namely they satisfy the inequality  $a = t_0 < t_2 < t_1 < t_3 = b$ . Additionally, we discuss the case  $\lambda_0 = \lambda_1$ , the function  $e^{\lambda_1 x}$  in (6) is replaced by  $x e^{\lambda_0 x}$ .

It follows from the above construction that the operator  $B_n$  defined by (5) satisfying (6) is a *positive* operator. Using a Korovkin-type theorem for extended Chebyshev systems we derive in Section 4 a sufficient criterion for the uniform convergence of  $B_n f$  on  $f \in C[a, b]$ . The criterion is formulated in terms of the basis functions  $p_{n,k}$  and their derivatives at the point  $b$ .

In Section 5 we consider exponential polynomials for equidistant eigenvalues  $\Lambda_n = (\lambda_0, \dots, \lambda_n)$ , i.e. for  $\lambda_j = \lambda_0 + j\omega$  for  $j = 0, \dots, n$ . We briefly discuss the relationship of a Bernstein-type theorem due to S. Morigi and M. Neamtu with our results.

## 2. BERNSTEIN BASES FOR COMPLEX EIGENVALUES

In order to give the reader more intuition about exponential polynomials we shall recall some elementary facts. In the case of pairwise different  $\lambda_j, j = 0, \dots, n$ , the space  $E_{(\lambda_0, \dots, \lambda_n)}$  is the linear span generated by the functions

$$e^{\lambda_0 x}, e^{\lambda_1 x}, \dots, e^{\lambda_n x}.$$

When some  $\lambda_j$  occurs  $m_j$  times in  $\Lambda_n = (\lambda_0, \dots, \lambda_n)$ , a basis for the space  $E_{(\lambda_0, \dots, \lambda_n)}$  is given by the linearly independent functions

$$x^s e^{\lambda_j x} \quad \text{for } s = 0, 1, \dots, m_j - 1.$$

We say that the vector  $\Lambda_n \in \mathbb{C}^{n+1}$  is *equivalent* to the vector  $\Lambda'_n \in \mathbb{C}^{n+1}$  if the corresponding differential operators are equal (so the spaces of solutions are equal). This is the same

to say that each  $\lambda$  occurs in  $\Lambda_n$  and  $\Lambda'_n$  with the same multiplicity. Since the differential operator  $L$  defined in (1) does not depend on the order of differentiation, it is clear that each permutation of the vector  $\Lambda_n$  is equivalent to  $\Lambda_n$ . Hence the space  $E_{(\lambda_0, \dots, \lambda_n)}$  does not depend on the order of the eigenvalues  $\lambda_0, \dots, \lambda_n$ .

The  $k$ -th derivative of a function  $f$  is denoted by  $f^{(k)}$ . A function  $f \in C^n(I, \mathbb{C})$  has a *zero of order  $k$  or of multiplicity  $k$*  at a point  $a \in I$  if  $f(a) = \dots = f^{(k-1)}(a) = 0$  and  $f^{(k)}(a) \neq 0$ . We shall repeatedly use the fact that

$$(7) \quad k! \cdot \lim_{x \rightarrow a} \frac{f(x)}{(x-a)^k} = f^{(k)}(a).$$

for any function  $f \in C^{(k)}(I)$  with  $f(a) = \dots = f^{(k-1)}(a) = 0$ .

**Definition 1.** A system of functions  $p_{n,k}$ ,  $k = 0, \dots, n$  in the space  $E_{(\lambda_0, \dots, \lambda_n)}$  is called Bernstein-like basis of  $E_{(\lambda_0, \dots, \lambda_n)}$  for  $a \neq b \in \mathbb{R}$  if and only if each function  $p_{n,k}$  has a zero of order  $k$  at  $a$  and a zero of order  $n-k$  at  $b$  for  $k = 0, \dots, n$ .

It is easy to see that a Bernstein-like basis  $p_{n,k}$ ,  $k = 0, \dots, n$  (if it exists) is indeed a basis for the space  $E_{(\lambda_0, \dots, \lambda_n)}$ . Moreover the basis functions are unique up to a non-zero multiplicative constant. In case of existence we shall require that

$$(8) \quad k! \lim_{x \rightarrow a, x > a} \frac{p_{n,k}(x)}{(x-a)^k} = p_{n,k}^{(k)}(a) = 1$$

and we shall call  $p_{n,k}$ ,  $k = 0, \dots, n$ , the *Bernstein basis* of  $E_{(\lambda_0, \dots, \lambda_n)}$  with respect to  $a \neq b$ .

In order to give a characterization of the existence of Bernstein bases, let us recall the general fact (cf. [26]) that for  $\Lambda_n = (\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$  there exists a unique function  $\Phi_n \in E_{(\lambda_0, \dots, \lambda_n)}$  such that  $\Phi_n(0) = \dots = \Phi_n^{(n-1)}(0) = 0$  and  $\Phi_n^{(n)}(0) = 1$ . An explicit formula for  $\Phi_n$  is given by

$$(9) \quad \Phi_n(x) := \Phi_{\Lambda_n}(x) := [\lambda_0, \dots, \lambda_n] e^{xz} = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{e^{xz}}{(z-\lambda_0) \dots (z-\lambda_n)} dz$$

where  $[\lambda_0, \dots, \lambda_n]$  denotes the divided difference, and  $\Gamma_r$  is a path in the complex plane defined by  $\Gamma_r(t) = re^{it}$ ,  $t \in [0, 2\pi]$ , surrounding all the scalars  $\lambda_0, \dots, \lambda_n$ . We shall call  $\Phi_n$  the *fundamental function*. Moreover we define

$$(10) \quad \Phi_{n,k}(x) := \det \begin{pmatrix} \Phi_n(x) & \dots & \Phi_n^{(k)}(x) \\ \vdots & & \vdots \\ \Phi_n^{(k)}(x) & \dots & \Phi_n^{(2k)}(x) \end{pmatrix}$$

for each  $k = 0, \dots, n$ . The following characterization is straightforward, see e.g. [19]:

**Theorem 2.** Let  $(\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$  and let  $a \neq b \in \mathbb{R}$ . Then the following statements are equivalent:

- a) There exists a Bernstein basis  $p_{n,k}$ ,  $k = 0, \dots, n$  in the space  $E_{(\lambda_0, \dots, \lambda_n)}$  for  $\{a, b\}$ ,

- b)  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for  $\{a, b\}$ ,  
c)  $\Phi_{n,k}(b-a) \neq 0$  for  $k = 0, \dots, n$ .

The equivalence of a) and b) also holds in the context of Chebyshev systems of real-valued functions, see [3], [10], [23], [24]. Further references on properties of Bernstein bases are [8] and [28].

It is well known that for *real eigenvalues*  $\lambda_0, \dots, \lambda_n$  the space  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system over *any* interval  $[a, b]$ , so a Bernstein basis exists in that case. In what follows we want to discuss the case of complex eigenvalues. The reader interested only in real eigenvalues may skip the rest of this section.

The following example is instructive:

**Example 3.** Let  $\Lambda_2 = (0, i, -i)$ . Then  $\Phi_2(x) = 1 - \cos x$  is the fundamental function. Since  $\Phi_2(2\pi k) = 0$  it follows that  $E_{\Lambda_2}$  does not possess a Bernstein basis for  $\{0, 2\pi k\}$  with  $k \in \mathbb{Z}$ . On the other hand, if  $b \neq 2\pi k, k \in \mathbb{Z}$ , then  $E_{\Lambda_2}$  does possess a Bernstein basis for  $\{0, b\}$ , explicitly given by

$$\begin{aligned} p_{2,2}(x) &= 1 - \cos x, \\ p_{2,1}(x) &= \sin x - \sin b \frac{1 - \cos x}{1 - \cos b}, \\ p_{2,0}(x) &= \frac{1 - \cos(x - b)}{1 - \cos b}. \end{aligned}$$

Note that  $E_{(0, i, -i)}$  possesses a Bernstein basis for  $\{0, \pi\}$  but that the subspace  $E_{(i, -i)}$  does not possess a Bernstein basis for  $\{0, \pi\}$  since  $\varphi_{(i, -i)}(x) = \sin x$  has then two zeros in  $\{0, \pi\}$ .

Let us take now  $b = 3\pi$  in Example 3. Then a Bernstein basis exists for  $\{0, 3\pi\}$  but the basis function  $p_{2,1}(x) = \sin x$  takes negative values on the interval  $[0, 3\pi]$ . So Bernstein basis functions may fail to be positive.

Let us recall that  $E_{(\lambda_0, \dots, \lambda_n)}$  is *closed under complex conjugation* if for each  $f \in E_{(\lambda_0, \dots, \lambda_n)}$  the complex conjugate function  $\bar{f}$  is again in  $E_{(\lambda_0, \dots, \lambda_n)}$ . It is easy to see that for complex numbers  $\lambda_0, \dots, \lambda_n$  the space  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation if and only if there exists a permutation  $\sigma$  of the indices  $\{0, \dots, n\}$  such that  $\bar{\lambda}_j = \lambda_{\sigma(j)}$  for  $j = 0, \dots, n$ . In other words,  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation if and only if the vector  $\Lambda_n = (\lambda_0, \dots, \lambda_n)$  is equivalent to the conjugate vector  $\bar{\Lambda}_n$ .

**Proposition 4.** Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for  $a \neq b \in \mathbb{R}$ . Then the space  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation if and only if the basis functions  $p_{n,k}$  are real-valued on  $\mathbb{R}$  for each  $k = 0, \dots, n$ .

*Proof.* Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation. Then  $\overline{p_{n,k}}$  is in  $E_{(\bar{\lambda}_0, \dots, \bar{\lambda}_n)} = E_{(\lambda_0, \dots, \lambda_n)}$ , it has a zero of order  $k$  at  $a$ , and a zero of order  $n - k$  at  $b$ . By uniqueness,  $\overline{p_{n,k}} = Dp_{n,k}$  for some constant  $D$ , and it follows from (8) that  $D = 1$ .

Thus  $p_{(\lambda_0, \lambda_1, \dots, \lambda_n), k}$  is real-valued. The converse is easy, since the  $p_{n, k}$  are real-valued functions on  $\mathbb{R}$ , and they form a basis.  $\square$

Assume  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation and  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for the closed interval  $[a, b]$ : then it is easy to see that the Bernstein basis functions are strictly positive on the interval  $(a, b)$ , and there exists  $x_0 \in (a, b)$  such that  $p_{n, k}$  is strictly increasing on  $(a, x_0)$  and decreasing on  $(x_0, b)$ , see e.g. [19]. Next we study when  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev over an interval  $[a, b]$ .

**Theorem 5.** *Let  $(\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$  and assume that  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation. Then the following statements are equivalent for  $a < b \in \mathbb{R}$ :*

- a)  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system over the interval  $[a, b]$ ,
- b)  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for all  $\{a, x\}$  with  $x \in (a, b]$ ,
- c) The functions  $x \mapsto \Phi_{n, k}(x - a)$ ,  $k = 0, \dots, n$ , have no zeros in  $(a, b]$ .

*Proof.* Clearly  $a) \rightarrow b)$  is trivial, and  $b)$  and  $c)$  are equivalent by Theorem 2. For  $c) \rightarrow a)$  note that the function  $\Phi_n$  is real-valued, so  $\Phi_n, \Phi'_n, \dots, \Phi_n^{(n)}$  are real-valued and they form a basis of  $E_{(\lambda_0, \dots, \lambda_n)}$ . We show that  $f \in E_{(\lambda_0, \dots, \lambda_n)}$  has at most  $n$  zeros in  $[a, b]$ . Since  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation we may assume that  $f$  is real-valued. We can write  $f = a_0 \Phi_n + \dots + a_n \Phi_n^{(n)}$ . Then  $f$  has a zero of order  $r \in \{0, \dots, n-1\}$  at  $a$ , implying that  $a_n = \dots = a_{n-r+1} = 0$ . Hence  $f$  is in the real linear span of  $\Phi_n, \dots, \Phi_n^{(n-r)}$ , which will be denoted by  $U_{n-r}$ . Since  $\Phi_{n, k}(b) \neq 0$  for  $k = 0, \dots, n-r$ , there exists by continuity some  $\delta > 0$  such that  $\Phi_{n, k}(y) \neq 0$  for all  $y \in [b - \delta, b + \delta]$  and  $k = 0, \dots, n-r$ . By Theorem 2.3 in [13, p. 52], applied to the open interval  $(a, b + \delta)$ , each function in  $U_r$  has at most  $n-r$  zeros (counting the multiplicities) in the open interval  $(a, b + \delta)$ . Hence  $f$  has at most  $n$  zeros on  $[a, b]$ .  $\square$

**Lemma 6.** *If  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system over  $[a, b]$  and  $\gamma$  is a real number, then  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system over  $[a + \gamma, b + \gamma]$ .*

**Lemma 7.** *If  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system over  $[a, b]$  and  $c$  is a complex number then  $E_{(\lambda_0 - c, \dots, \lambda_n - c)}$  is an extended Chebyshev system over  $[a, b]$ .*

*Proof.* If  $f \in E_{(\lambda_0, \dots, \lambda_n)}$ , then  $g$  defined by  $g(x) = e^{-cx} f(x)$  is in  $E_{(\lambda_0 - c, \dots, \lambda_n - c)}$ . If  $g$  had more than  $n$  zeros in  $[a, b]$  then  $f$  would have more than  $n$  zeros in  $[a, b]$ , a contradiction.  $\square$

**Lemma 8.** *If  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system over  $[a, b]$  and  $c$  is a positive number, then  $E_{(\lambda_0 c, \dots, \lambda_n c)}$  is an extended Chebyshev system over  $[a, a + \frac{b-a}{c}]$ .*

*Proof.* By Lemma 6 we may assume that  $a = 0$ . If  $f \in E_{(\lambda_0, \dots, \lambda_n)}$  then  $g$ , defined by  $g(x) := f(cx)$ , is in  $E_{(\lambda_0 c, \dots, \lambda_n c)}$ . Suppose that  $g$  has more than  $n$  zeros in  $[0, \frac{b}{c}]$ . Then  $f$  has more than  $n$  zeros in  $[0, b]$ , a contradiction.  $\square$

The following is the main result of this section:

**Theorem 9.** *Let  $(\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$  and assume that  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation. If  $|\operatorname{Im} \lambda_j| \leq M_n$  for  $j = 0, \dots, n$ , then  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for the interval  $[a, b]$ , provided  $b - a < \pi/M_n$ .*

*Proof.* By an inductive argument, it suffices to prove the following two statements for an extended Chebyshev system  $E_{(\lambda_0, \dots, \lambda_n)}$  over  $[a, b]$ , closed under complex conjugation:

- 1) If  $\lambda_{n+1}$  is real then  $E_{(\lambda_0, \dots, \lambda_{n+1})}$  is an extended Chebyshev system for  $[a, b]$ ,
- 2) If  $\lambda_{n+1}$  is a non-real complex number, then  $E_{(\lambda_0, \dots, \lambda_n, \lambda_{n+1}, \overline{\lambda_{n+1}})}$  is an extended Chebyshev system over  $[a, d]$ , for any  $d$  with  $a < d \leq b$  and  $d - a < \frac{\pi}{|\operatorname{Im} \lambda_{n+1}|}$ .

For a proof of 1) we use a standard argument: let  $f \in E_{(\lambda_0, \dots, \lambda_{n+1})}$  be non-zero with  $m$  zeros in  $[a, b]$ . We may assume that  $f$  is real-valued since  $E_{(\lambda_0, \dots, \lambda_{n+1})}$  is closed under complex conjugation. Then  $h(x) := e^{-\lambda_{n+1}x} f(x)$  is real-valued and it has  $m$  zeros in  $[a, b]$ . By Rolle's theorem  $h'(x)$  has at least  $m - 1$  zeros in  $[a, b]$ . Since

$$e^{\lambda_{n+1}x} h'(x) = e^{\lambda_{n+1}x} \frac{d}{dx} (e^{-\lambda_{n+1}x} f(x)) = \left( \frac{d}{dx} - \lambda_{n+1} \right) f(x) =: F(x)$$

we conclude that  $F$  has at least  $m - 1$  zeros in  $[a, b]$ . But  $F$  is in  $E_{(\lambda_0, \dots, \lambda_n)}$ , so it has at most  $n$  zeros, and hence  $m - 1 \leq n$ .

For a proof of 2) note that by Lemma 7 we may assume that  $c := \operatorname{Re} \lambda_{n+1}$  is zero. Without loss of generality let  $\operatorname{Im} \lambda_{n+1} > 0$ , and by Lemma 8 it suffices to prove 2) for the case that  $\lambda_{n+1} = i$ . It is clear that  $E_{(\lambda_0, \dots, \lambda_n, i, -i)}$  is closed under complex conjugation. Furthermore, by Lemma 6 we may assume that  $[a, d] \subset I := (-\frac{1}{2}\pi, \frac{1}{2}\pi)$ . Let us introduce the auxiliary function

$$v(x) = \frac{1 + \tan x}{1 - \tan x}$$

defined on the interval  $I$ . A computation shows that  $v' = v^2 + 1$ . Let  $u$  be a primitive function of  $v$ , so we have  $u' = v$  and  $u'' = (u')^2 + 1$ . Let us define  $g := e^u$ . Then  $g$  satisfies the differential equation  $g''g - 2(g')^2 = g^2$  and a computation shows that

$$g \frac{d}{dx} \left[ g^{-2} \frac{d}{dx} (gf) \right] = \left( \frac{d^2}{dx^2} + 1 \right) f.$$

Now we can argue as above: if  $f \in E_{(\lambda_0, \dots, \lambda_n, i, -i)}$  has  $m$  zeros in the interval  $[a, d]$ , so does  $gf$ . Thus  $\frac{d}{dx} (gf)$  has at least  $m - 1$  zeros in  $[a, d]$ . Hence  $g^{-2} \frac{d}{dx} (gf)$  has at least  $m - 1$  zeros in  $[a, d]$  and we conclude that  $\frac{d}{dx} \left[ g^{-2} \frac{d}{dx} (gf) \right]$  has at least  $m - 2$  zeros in  $[a, d]$ . Therefore  $\left( \frac{d^2}{dx^2} + 1 \right) f$  has at least  $m - 2$  zeros in  $[a, d]$ . Since  $d \leq b$  and  $\left( \frac{d^2}{dx^2} + 1 \right) f \in E_{(\lambda_0, \dots, \lambda_n)}$ , and since  $E_{(\lambda_0, \dots, \lambda_n)}$  is a Chebyshev system over  $[a, b]$  we obtain  $m - 2 \leq n$ .  $\square$

For a discussion of complex zeros of exponential polynomials we refer to the recent work [32].

## 3. RECURSIVE RELATIONS FOR BERNSTEIN BASES

Let  $(\lambda_0, \dots, \lambda_n) \in \mathbb{C}^{n+1}$  and assume that  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for  $\{a, b\}$ . We can construct a Bernstein basis for  $a \neq b \in \mathbb{R}$  via the following procedure: put  $q_0(x) = \Phi_n(x - a)$ , which clearly has a zero of order  $n$  at  $a$ . Then  $q_0(b) \neq 0$  since  $E_{(\lambda_0, \dots, \lambda_n)}$  is an extended Chebyshev system for  $\{a, b\}$ . We define  $q_1 := q_0^{(1)} - \alpha_0 q_0$ , where  $\alpha_0 = q_0^{(1)}(b)/q_0(b)$ . Then  $q_1$  has a zero of order  $n - 1$  at  $a$  and a zero of order 1 at  $b$ . For  $k \geq 2$  we define  $q_k$  recursively by

$$(11) \quad q_k := q_{k-1}^{(1)} - (\alpha_{k-1} - \alpha_{k-2}) \cdot q_{k-1} - \beta_k q_{k-2}$$

with coefficients  $\alpha_{k-1}, \alpha_{k-2}$  and  $\beta_k$  to be determined. Note that  $q_k$  has a zero of order at least  $k - 2$  at  $b$ , and a zero of order  $n - k$  at  $a$ . The coefficients  $\alpha_{k-1}, \alpha_{k-2}$  and  $\beta_k$  are chosen so that  $q_k$  has a zero of order  $k$  at  $b$ , which is achieved by defining

$$(12) \quad \beta_k := \frac{q_{k-1}^{(k-1)}(b)}{q_{k-2}^{(k-2)}(b)} \text{ and } \alpha_{k-1} := \frac{q_{k-1}^{(k)}(b)}{q_{k-1}^{(k-1)}(b)}.$$

Then  $p_{n,n-k} := q_k$  for  $k = 0, \dots, n$  is the Bernstein basis satisfying condition (8).

The proof of the following proposition is easy and therefore omitted.

**Proposition 10.** *Let  $c \in \mathbb{C}$  and define  $c + \Lambda_n := (c + \lambda_0, \dots, c + \lambda_n)$ . If there exists a Bernstein basis  $p_{n,k}, k = 0, \dots, n$  for  $E_{\Lambda_n}$  and  $a \neq b$ , then there exists a Bernstein basis of  $E_{c+\Lambda_n}$  given by*

$$p_{c+\Lambda_n, k}(x) = p_{\Lambda_n, k}(x) e^{c(x-a)}$$

for  $k = 0, \dots, n$ .

In the polynomial case the Bernstein basis  $P_{n,k}$  defined in (2) satisfies the useful identity

$$\frac{d}{dx} P_{n,k} = P_{n-1, k-1} - \frac{n-k}{b-a} P_{n-1, k},$$

which follows directly by differentiating (2). Next we present its analog for exponential polynomials. In what follows we shall use the more precise but lengthier notation  $p_{(\lambda_0, \dots, \lambda_n), k}$  instead of  $p_{n,k}$ .

**Proposition 11.** *Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$  and  $E_{(\lambda_0, \dots, \lambda_{n-1})}$  are extended Chebyshev systems for  $a \neq b \in \mathbb{R}$ . Define for  $k = 0, \dots, n - 1$  the numbers*

$$(13) \quad d_k := \lim_{x \uparrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), k}(x)}{p_{(\lambda_0, \dots, \lambda_{n-1}), k}(x)} \neq 0.$$

Then,

$$(14) \quad \left( \frac{d}{dx} - \lambda_n \right) p_{(\lambda_0, \dots, \lambda_n), k} = p_{(\lambda_0, \dots, \lambda_{n-1}), k-1} + d_k p_{(\lambda_0, \dots, \lambda_{n-1}), k}$$



for any  $k = 1, \dots, n-1$ . Furthermore, for  $k = 0$  the right hand side of (14) is equal to  $d_0 p_{(\lambda_0, \dots, \lambda_{n-1}), 0}$ , while for  $k = n$ , it is equal to  $p_{(\lambda_0, \dots, \lambda_{n-1}), n-1}$ .

*Proof.* Let  $f_k$  be the left hand side of (14) and let  $1 \leq k \leq n-1$ . Using the fact that  $f_k$  has a zero of order  $k-1$  at  $a$  and a zero of order  $n-k-1$  at  $b$ , it is easy to see that  $f_k = c_k p_{(\lambda_0, \dots, \lambda_{n-1}), k-1} + d_k p_{(\lambda_0, \dots, \lambda_{n-1}), k}$  for some constants  $c_k$  and  $d_k$ . Now simple limit considerations complete the proof.  $\square$

**Proposition 12.** *Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$  and  $E_{(\lambda_0, \dots, \lambda_{n-1})}$  are extended Chebyshev systems for  $a \neq b \in \mathbb{R}$ . Assume that  $p_{(\lambda_0, \dots, \lambda_{n-1}, \lambda_n), k}(x) = p_{(\lambda_0, \dots, \lambda_{n-1}, \eta_n), k}(x)$  on  $(a, b)$  for a given  $k$ . Then  $\lambda_n = \eta_n$ . The same holds if instead of  $\lambda_n$  we consider any other eigenvalue.*

*Proof.* Suppose first that  $1 \leq k \leq n-1$ . By Proposition 11 we obtain the equations

$$\begin{aligned} \left( \frac{d}{dx} - \lambda_n \right) p_{(\lambda_0, \dots, \lambda_{n-1}, \lambda_n), k} &= p_{(\lambda_0, \dots, \lambda_{n-1}), k-1} + d_k p_{(\lambda_0, \dots, \lambda_{n-1}), k}, \\ \left( \frac{d}{dx} - \eta_n \right) p_{(\lambda_0, \dots, \lambda_{n-1}, \eta_n), k} &= p_{(\lambda_0, \dots, \lambda_{n-1}), k-1} + D_k p_{(\lambda_0, \dots, \lambda_{n-1}), k}. \end{aligned}$$

Using the order of the zeros at  $b$  and assumption  $p_{(\lambda_0, \dots, \lambda_{n-1}, \lambda_n), k}(x) = p_{(\lambda_0, \dots, \lambda_{n-1}, \eta_n), k}(x)$  we get  $d_k = D_k$ , so the right hand sides are equal. Subtracting we conclude that  $\lambda_n = \eta_n$ . The cases  $k = 0$  and  $k = 1$  immediately follow from Proposition 11.  $\square$

For the polynomial Bernstein basis over the interval  $[0, 1]$  one often uses identities like

$$(15) \quad 1 = (x + (1-x))^n = \sum_{k=0}^n \frac{n!}{(n-k)!} P_{n,k}(x)$$

where  $P_{n,k}$  is defined as in (2). The following is an analog for exponential polynomials:

**Theorem 13.** *Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$  and  $E_{(\lambda_0, \dots, \lambda_{n-1})}$  are extended Chebyshev systems for  $a \neq b \in \mathbb{R}$ . Let  $d_0, \dots, d_{n-1}$  be the non-zero numbers defined in (13). Then*

$$(16) \quad e^{(x-a)\lambda_n} = p_{(\lambda_0, \dots, \lambda_n), 0}(x) + \sum_{k=1}^n (-1)^k d_0 \cdots d_{k-1} \cdot p_{(\lambda_0, \dots, \lambda_n), k}(x).$$

Furthermore for  $k = 1, \dots, n-1$ , we have the equality

$$(17) \quad d_0 \cdots d_{k-1} = (-1)^n \frac{e^{(b-a)\lambda_n}}{p_{(\lambda_0, \dots, \lambda_n), n}(b)} \frac{1}{d_k \cdots d_{n-1}}.$$

*Proof.* Write  $e^{(x-a)\lambda_n} = \sum_{k=0}^n \beta_k p_{(\lambda_0, \dots, \lambda_n), k}(x)$  for coefficients  $\beta_0, \dots, \beta_n$ . Inserting  $x = a$  yields  $1 = \beta_0 p_{(\lambda_0, \dots, \lambda_n), 0}(0)$ , so  $\beta_0 = 1$  by (8). Proposition 11 yields

$$\begin{aligned} 0 &= \left( \frac{d}{dx} - \lambda_n \right) e^{(x-a)\lambda_n} = d_0 p_{(\lambda_0, \dots, \lambda_{n-1}), 0}(x) + \beta_1 p_{(\lambda_0, \dots, \lambda_{n-1}), 0}(x) \\ &\quad + \sum_{k=1}^{n-2} (\beta_{k+1} + \beta_k d_k) p_{(\lambda_0, \dots, \lambda_{n-1}), k}(x) + p_{(\lambda_0, \dots, \lambda_{n-1}), n-1}(x) [\beta_{n-1} d_{n-1} + \beta_n]. \end{aligned}$$

Thus  $\beta_1 = -d_0$ ,  $\beta_{k+1} = -\beta_k d_k$  for  $k = 1, \dots, n-2$ , and  $\beta_{n-1} d_{n-1} + \beta_n = 0$ . Hence, for  $k = 1, \dots, n$  we have  $\beta_k = (-1)^k d_0 \cdots d_{k-1}$ , and then (16) follows. Furthermore, by inserting  $x = b$  in (16) and recalling that  $p_{(\lambda_0, \dots, \lambda_n), k}$  has a zero of order  $n - k$  at  $x = b$ , we see that

$$(-1)^n d_0 \cdots d_{n-1} \cdot p_{(\lambda_0, \dots, \lambda_n), n}(b) = e^{(b-a)\lambda_n}.$$

Thus, we get (17).  $\square$

Theorem 13 does not hold when the assumption of having an extended Chebyshev system  $E_{(\lambda_0, \dots, \lambda_{n-1})}$  for  $a \neq b$  is dropped: in Example 3, with  $b = \pi$ , one has that

$$1 = \frac{1}{2} (1 - \cos x) + 0 \cdot \sin x + \frac{1}{2} (1 + \cos x),$$

so 1 is a linear combination of the Bernstein basis functions with a *zero* coefficient.

**Theorem 14.** *Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$ ,  $E_{(\lambda_0, \lambda_2, \dots, \lambda_n)}$ ,  $E_{(\lambda_1, \dots, \lambda_n)}$  and  $E_{(\lambda_2, \dots, \lambda_n)}$  are extended Chebyshev systems for  $a \neq b \in \mathbb{R}$ . Let  $\lambda_0 \neq \lambda_1$ . Then there exists a constant  $C_k^{\lambda_0, \lambda_1}(\Lambda_n) \neq 0$  such that*

$$(18) \quad p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k} - p_{(\lambda_1, \lambda_2, \dots, \lambda_n), k} = C_k^{\lambda_0, \lambda_1}(\Lambda_n) \cdot p_{(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n), k+1}.$$

Moreover,

$$(19) \quad \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k}(x)}{p_{(\lambda_1, \dots, \lambda_n), k}(x)} \neq 1.$$

*Proof.* Let  $B(x)$  be the function on the left hand side of (18). Then  $B$  has a zero of order  $k+1$  at  $a$ , since  $p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k}$  and  $p_{(\lambda_1, \lambda_2, \dots, \lambda_n), k}$  have a zero of order  $k$  at  $a$ , and  $B^{(k)}(a) = \lim_{x \rightarrow a} \frac{B(x)}{(x-a)^k} = 0$  by (8). Furthermore  $B$  has a zero of order  $n - k - 1$  at  $b$ . By Proposition 12,  $B$  is not identically zero (here we need that  $E_{(\lambda_2, \dots, \lambda_n)}$  is an extended Chebyshev system for  $a \neq b$ ). Since  $B \in E_{(\lambda_0, \lambda_1, \dots, \lambda_n)}$ , it must be a non-zero multiple of  $p_{(\lambda_0, \lambda_1, \dots, \lambda_n), k+1}$ .

Finally, suppose that the limit in (19) is equal to 1. Then  $p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k}^{(n-1-k)}(b) = p_{(\lambda_1, \dots, \lambda_n), k}^{(n-1-k)}(b)$ . By (18) we conclude that  $p_{(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n), k+1}^{(n-1-k)}(b) = 0$ . Hence  $p_{(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n), k+1}$  has a zero of order  $n - k$  at  $b$  and a zero of order  $k+1$  at  $a$ , a contradiction.  $\square$

In the case of equidistant eigenvalues it is possible to define a Bernstein basis explicitly:

**Proposition 15.** *Suppose that  $\omega \neq 0$  and  $\lambda_j = \lambda_0 + j\omega$  for  $j = 0, \dots, n$ . Then*

$$(20) \quad p_{n,k}(x) := \frac{e^{\lambda_0(x-a)}}{k!\omega^k} (e^{\omega(x-a)} - 1)^k \left( \frac{1 - e^{\omega(x-b)}}{1 - e^{\omega(a-b)}} \right)^{n-k}$$

*is a Bernstein basis for  $[a, b]$  satisfying*

$$(21) \quad k! \lim_{x \rightarrow a} p_{n,k}(x) / (x - a)^k = 1.$$

*Proof.* It is easy to see that  $p_{n,k}$  is an exponential polynomial. Furthermore,  $p_{n,k}$  has a zero at  $x = a$  of order  $k$  and a zero of order  $n - k$  at  $x = b$ .  $\square$

**Lemma 16.** *Let  $\omega \neq 0$  and let  $\lambda_j = \lambda_0 + j\omega$  for  $j = 0, \dots, n$ . Then*

$$p_{(\lambda_0, \lambda_1, \dots, \lambda_{n-1}), k} - p_{(\lambda_1, \lambda_2, \dots, \lambda_n), k} = -(k+1)\omega p_{(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n), k+1}.$$

*Proof.* This is a computation using (20).  $\square$

The following result is crucial for the proof of the existence of a Bernstein operator. In this theorem we shall use a homotopy argument for the eigenvalues  $\Lambda_n$  and the assumption (22) will guarantee that the corresponding Bernstein bases with respect to the points  $a \neq b$  exist.

**Theorem 17.** *Suppose that  $E_{(\lambda_2, \dots, \lambda_n)}$  is closed under complex conjugation and  $0 < b - a < \pi/M_n$ , where*

$$(22) \quad M_n = \max \{ |Im \lambda_j| : \text{for } j = 2, \dots, n \}.$$

*If  $\lambda_0, \lambda_1 \in \mathbb{R}$  and  $\lambda_0 < \lambda_1$ , then*

$$(23) \quad \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k}(x)}{p_{(\lambda_1, \dots, \lambda_n), k}(x)} < 1.$$

*Furthermore, the function of  $\lambda \in \mathbb{R}$*

$$(24) \quad \lambda \mapsto p_{(\lambda, \lambda_2, \dots, \lambda_n), k}(x)$$

*is strictly increasing for each  $x \in (a, b)$ .*

*Proof.* By Proposition 4,  $p_{(\lambda, \lambda_2, \dots, \lambda_n), k}(x)$  is real valued for every real  $\lambda$ . Let now  $\lambda_0 < \lambda_1$  be real. It follows that  $C_k^{\lambda_0, \lambda_1}(\Lambda_n)$  in (18) is real. Clearly, for  $\lambda_0 < \lambda_1$  and fixed  $(\lambda_2, \dots, \lambda_n)$  the function in (24) is increasing if  $C_k^{\lambda_0, \lambda_1}(\Lambda_n)$  in (18) is negative. From the inductive formula (11) we get that the function  $(\lambda_0, \dots, \lambda_n) \mapsto p_{(\lambda_0, \dots, \lambda_n), k}$  is continuous. Now (18) implies that  $(\lambda_0, \dots, \lambda_n) \mapsto C_k^{\lambda_0, \lambda_1}(\Lambda_n)$  is continuous. For  $\lambda_0 < \lambda_1$ , define  $\mu_j := \lambda_0 + j(\lambda_1 - \lambda_0)/n$ , where  $j = 0, \dots, n$ . Since  $\mu_0 = \lambda_0$  and  $\mu_n = \lambda_1$  the function

$$t \mapsto C_k^{t\lambda_0 + (1-t)\mu_0, t\lambda_1 + (1-t)\mu_n}(t(\lambda_0, \dots, \lambda_n) + (1-t)(\mu_0, \dots, \mu_n))$$

is continuous, and by Theorem 14, it has no zero on  $\mathbb{R}$ . It follows that this function must have constant sign. Hence it suffices to show that  $C_k^{\mu_0, \mu_n}(\mu_0, \dots, \mu_n) < 0$ . But this follows from Lemma 16. Thus,  $\lambda \mapsto p_{(\lambda, \lambda_2, \dots, \lambda_n), k}(x)$  is increasing, so for  $a < x < b$  and  $\lambda_0 < \lambda_1$ ,

$$(25) \quad p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k}(x) < p_{(\lambda_1, \lambda_2, \dots, \lambda_n), k}(x).$$

Dividing (25) by its right hand side, using (19), and taking the limit  $x \uparrow b$ , we get (23).  $\square$

**Example 18.** Let us take  $\lambda \in \mathbb{R}$ . Then the fundamental function  $\Phi_{(\lambda, i, -i)}$  with respect to  $(\lambda, i, -i)$  is given by

$$\Phi_{(\lambda, i, -i)}(x) = \frac{e^{\lambda x} - \cos x - \lambda \sin x}{\lambda^2 + 1}.$$

Since  $\Phi_{(\lambda, i, -i)}$  is equal to the basis function  $p_{(\lambda, i, -i), 2}$  it follows that  $\lambda \mapsto \Phi_{(\lambda, i, -i)}(x)$  is increasing for any  $x$  in the interval  $(0, \pi)$ . Differentiating, it is easy to check that  $\lambda \mapsto \Phi_\lambda(x)$  is decreasing whenever  $x < 0$  is small in absolute value, and  $\lambda$  is sufficiently large. For  $x$  with  $\pi < x < 2\pi$  it might be checked that  $\lambda \mapsto \Phi_\lambda(x)$  is not increasing.

#### 4. CONSTRUCTION OF THE BERNSTEIN OPERATOR

We now proceed to our first main result which roughly says the following: given two functions  $e^{\lambda_0 x}$  and  $e^{\lambda_1 x}$  in the extended Chebyshev system  $E_{(\lambda_0, \dots, \lambda_n)}$  over  $[a, b]$  we can find points  $t_0, \dots, t_n$  in the interval  $[a, b]$  and positive numbers  $\alpha_0, \dots, \alpha_n$  such that the operator  $B_n$  defined by (27) below reproduces (or preserves) the functions  $e^{\lambda_0 x}$  and  $e^{\lambda_1 x}$ , i.e. (28) holds.

**Theorem 19.** Let  $\lambda_0, \dots, \lambda_n$  be complex numbers with  $\lambda_0$  and  $\lambda_1$  real and  $\lambda_0 < \lambda_1$ . Suppose  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation and  $0 < b - a < \pi/M_n$ , where

$$M_n := \max \{ |\operatorname{Im} \lambda_j| : j = 0, \dots, n \}.$$

Define inductively points  $t_0, \dots, t_n$  by setting  $t_0 = a$  and

$$e^{(\lambda_0 - \lambda_1)(t_k - t_{k-1})} = \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k-1}(x)}{p_{(\lambda_1, \dots, \lambda_n), k-1}(x)}$$

for  $k = 1, 2, \dots, n$ . Then

$$a = t_0 < t_1 < \dots < t_n = b.$$

Put  $\alpha_0 = 1$ , and define numbers

$$(26) \quad \alpha_k = e^{-\lambda_0(t_k - a)} (-1)^k \prod_{l=0}^{k-1} \lim_{x \rightarrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), l}(x)}{p_{(\lambda_1, \dots, \lambda_n), l}(x)}$$

for  $k = 1, \dots, n$ . Then  $\alpha_0, \dots, \alpha_n > 0$  and the operator  $B_{(\lambda_0, \dots, \lambda_n)}$  on  $C[a, b]$  defined by

$$(27) \quad B_{(\lambda_0, \dots, \lambda_n)} f = \sum_{k=0}^n \alpha_k f(t_k) p_{(\lambda_0, \dots, \lambda_n), k}$$

fixes the functions  $e^{\lambda_0 x}$  and  $e^{\lambda_1 x}$ , i.e.

$$(28) \quad B_{(\lambda_0, \dots, \lambda_n)}(e^{\lambda_0 x}) = e^{\lambda_0 x} \text{ and } B_{(\lambda_0, \dots, \lambda_n)}(e^{\lambda_1 x}) = e^{\lambda_1 x}.$$

Moreover, the real numbers  $t_0, \dots, t_n$  and  $\alpha_0, \dots, \alpha_n$  satisfying (28) are unique.

*Proof.* First let us only assume that  $E_{(\lambda_0, \lambda_1, \dots, \lambda_n)}$ ,  $E_{(\lambda_1, \lambda_2, \dots, \lambda_n)}$  and  $E_{(\lambda_0, \lambda_2, \dots, \lambda_n)}$  are extended Chebyshev systems over  $\{a, b\}$ , in order to clarify where we need the assumption  $0 < b - a < \pi/M_n$ . Let  $\beta_0, \dots, \beta_n$  and  $\gamma_0, \dots, \gamma_n$  be the unique *non-zero* coefficients, found in Theorem 13, that satisfy

$$(29) \quad e^{\lambda_0(x-a)} = \sum_{k=0}^n \beta_k p_{(\lambda_0, \dots, \lambda_n), k}(x) \text{ and } e^{\lambda_1(x-a)} = \sum_{k=0}^n \gamma_k p_{(\lambda_0, \dots, \lambda_n), k}(x).$$

The reproducing property of the Bernstein operator for  $e^{\lambda_0 x}$  in (28) implies that

$$\sum_{k=0}^n e^{\lambda_0(t_k-a)} \alpha_k p_{(\lambda_0, \dots, \lambda_n), k}(x) = \sum_{k=0}^n \beta_k p_{(\lambda_0, \dots, \lambda_n), k}(x).$$

Since  $p_{(\lambda_0, \dots, \lambda_n), k}$  is a basis we conclude that  $e^{\lambda_0(t_k-a)} \alpha_k = \beta_k$ . Similarly,  $e^{\lambda_1(t_k-a)} \alpha_k = \gamma_k$  follows from  $B_{(\lambda_0, \dots, \lambda_n)}(e^{\lambda_1 x}) = e^{\lambda_1 x}$ . Dividing, we see that  $t_k$  satisfies the equation

$$(30) \quad e^{(\lambda_0 - \lambda_1)t_k} = \frac{\beta_k}{\gamma_k} e^{(\lambda_0 - \lambda_1)a}.$$

Hence, for  $\alpha_k$  we obtain

$$(31) \quad \alpha_k = e^{-\lambda_0(t_k-a)} \beta_k.$$

It is easy to see that  $\beta_k$  and  $\gamma_k$  are real, since the functions  $p_{(\lambda_0, \dots, \lambda_n), k}$  are real-valued (Proposition 4) and both  $\lambda_0$  and  $\lambda_1$  are real. Now  $t \mapsto e^{(\lambda_0 - \lambda_1)t}$  is real-valued and injective, so  $t_k$  is *uniquely* determined by (30), and hence so is  $\alpha_k$  by (31). Moreover, it is easy to see that the points  $t_0 = a$  and  $t_n = b$  satisfy (30).

Next we want to show that  $\alpha_k$  is positive: by Theorem 13 (applied to  $\lambda_0$  instead of  $\lambda_n$ ) we have  $\beta_k = (-1)^k \tilde{d}_0 \dots \tilde{d}_{k-1}$ , where  $\tilde{d}_l$  is given by

$$(32) \quad \tilde{d}_l = \lim_{x \rightarrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), l}(x)}{p_{(\lambda_1, \dots, \lambda_n), l}(x)}.$$

The positivity of  $p_{(\lambda_1, \dots, \lambda_n), k}(x)$  on  $(a, b)$  implies that  $\tilde{d}_l$  is negative. Hence, for  $k = 1, \dots, n-1$ , equation (31) yields

$$(33) \quad \alpha_k = e^{-\lambda_0(t_k-a)} \beta_k = e^{-\lambda_0(t_k-a)} (-1)^k \tilde{d}_0 \dots \tilde{d}_{k-1}$$

showing that  $\alpha_k$  is positive. Similarly,  $\gamma_k = (-1)^k D_0 \dots D_{k-1}$  where  $D_l$  is given by

$$(34) \quad D_l = \lim_{x \rightarrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), l}(x)}{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), l}(x)}.$$

Thus the points  $t_k$  are defined by

$$(35) \quad e^{(\lambda_0 - \lambda_1)t_k} = \frac{\beta_k}{\gamma_k} e^{(\lambda_0 - \lambda_1)a} = \frac{\tilde{d}_0 \dots \tilde{d}_{k-1}}{D_0 \dots D_{k-1}} e^{(\lambda_0 - \lambda_1)a}.$$

Note that for  $k = 1, \dots, n$

$$(36) \quad e^{(\lambda_0 - \lambda_1)(t_k - t_{k-1})} = \frac{e^{(\lambda_0 - \lambda_1)t_k}}{e^{(\lambda_0 - \lambda_1)t_{k-1}}} = \frac{\tilde{d}_{k-1}}{D_{k-1}} = \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k-1}(x)}{p_{(\lambda_1, \dots, \lambda_n), k-1}(x)}.$$

Next we show that  $t_k$  is in the interval  $[a, b]$ . Since  $t_0 = a$  and  $t_n = b$ , it suffices to show that  $t_{k-1} < t_k$ . Since  $\lambda_0 < \lambda_1$  the requirement  $t_{k-1} < t_k$  is equivalent to the requirement that

$$\lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k-1}(x)}{p_{(\lambda_1, \dots, \lambda_n), k-1}(x)} < 1.$$

Theorem 17 tells us this is true under the assumption that  $|b - a| < \pi/M_n$ , thus finishing the proof.  $\square$

In Example 18 we have computed the fundamental function for  $(\lambda, i, -i)$  and it is easy to see that the Bernstein basis function  $p_{(\lambda, i, -i), 1}$  for  $\{0, b\}$  is given by

$$(37) \quad p_{(\lambda, i, -i), 1}(x) = \frac{\lambda e^{\lambda x} + \sin x - \lambda \cos x}{\lambda^2 + 1} - \frac{(e^{\lambda x} - \cos x - \lambda \sin x)(\lambda e^{\lambda b} + \sin b - \lambda \cos b)}{(e^{\lambda b} - \cos b - \lambda \sin b)(\lambda^2 + 1)}.$$

Simple computations show that  $p'_{(\lambda, i, -i), 1}(b) = \frac{e^{\lambda b} \cos b - \lambda e^{\lambda b} \sin b - 1}{e^{\lambda b} - \cos b - \lambda \sin b}$  and

$$(38) \quad \frac{p'_{(-\lambda, i, -i), 1}(b)}{p'_{(\lambda, i, -i), 1}(b)} = \left( \frac{e^{\lambda b} - \cos b - \lambda \sin b}{1 - e^{\lambda b} \cos b + \lambda e^{\lambda b} \sin b} \right)^2.$$

Consider now the Bernstein operator for  $\Lambda_3 = (-1, 1, i, -i)$  for the interval  $[0, 3.5]$ . Using Theorem 2 it can be seen that  $E_{(1, i, -i)}$  and  $E_{(-1, i, -i)}$  are extended Chebyshev systems, at least for the interval  $[0, 3.8]$ . By property 1) in the proof of Theorem 9,  $E_{(-1, 1, i, -i)}$  is an extended Chebyshev system for  $[0, 3.8]$ . Hence by the proof of Theorem 13 it follows from (36) that

$$e^{-2(t_2 - t_1)} = \frac{p'_{(-1, i, -i), 1}(3.5)}{p'_{(1, i, -i), 1}(3.5)} \approx 2.8454 > 1,$$

so  $t_2 - t_1$  must be *negative*, and thus the Bernstein operator for  $\Lambda_3 = (-1, 1, i, -i)$  has the property that  $t_0 < t_2 < t_1 < t_3$ .

If we take  $b = \pi$  then (37) shows that  $p_{(\lambda, i, -i), 1} = \sin x$ . Thus Theorem 14 is not valid if we drop the assumption that  $E_{(\lambda_2, \dots, \lambda_n)}$  is an extended Chebyshev system for  $a \neq b \in \mathbb{R}$ .

By a limiting process one can handle the case that the eigenvalues  $\lambda_0, \lambda_1$  are equal when replacing  $e^{\lambda_1 x}$  by  $x e^{\lambda_0 x}$  in (28). However, we have not been able to show that in this case the nodes  $a = t_0 \leq t_1 \leq \dots \leq t_n = b$  are distinct.

**Theorem 20.** *Let  $\lambda_0, \dots, \lambda_n$  be complex numbers such that  $\lambda_0 = \lambda_1$  is real. Suppose that  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation and that  $0 < b - a < \pi/M_n$  for*

$$M_n := \max \{ |Im \lambda_j| : j = 0, \dots, n \}.$$

*Then there exist unique nodes  $a = t_0 \leq t_1 \leq \dots \leq t_n = b$  and unique positive numbers  $\alpha_0, \dots, \alpha_n$  such that the operator defined for  $f \in C[a, b]$  by*

$$(39) \quad B_{(\lambda_0, \dots, \lambda_n)} f = \sum_{k=0}^n \alpha_k f(t_k) p_{(\lambda_0, \dots, \lambda_n), k}$$

*fixes the functions  $e^{\lambda_0 x}$  and  $x e^{\lambda_0 x}$ .*

*Proof.* For  $\varepsilon \geq 0$  define  $\Lambda_\varepsilon := (\lambda_0, \lambda_0 + \varepsilon, \lambda_2, \dots, \lambda_n)$ . By Theorem 19 there exist for each  $\varepsilon > 0$  points  $a = t_0(\varepsilon) < t_1(\varepsilon) < \dots < t_n(\varepsilon) = b$  and positive numbers  $\alpha_0(\varepsilon), \dots, \alpha_n(\varepsilon)$  such that the corresponding Bernstein operator  $B_{\Lambda_\varepsilon}$  satisfies

$$B_{\Lambda_\varepsilon}(e^{\lambda_0 x}) = e^{\lambda_0 x} \text{ and } B_{\Lambda_\varepsilon}(e^{(\lambda_0 + \varepsilon)x}) = e^{(\lambda_0 + \varepsilon)x}.$$

By compactness of the interval  $[a, b]$  there exists a sequence of positive numbers  $\varepsilon_m \rightarrow 0$  such that  $t_j(\varepsilon_m)$  converges to numbers  $t_j$  for  $m \rightarrow \infty$  and for each  $j = 0, \dots, n$ . Clearly one has  $a = t_0 \leq t_1 \leq \dots \leq t_n = b$ . Let us write  $e^{\lambda_0(x-a)} = \sum_{k=0}^n \beta_k(\varepsilon) p_{\Lambda_\varepsilon, k}(x)$  for  $\varepsilon \geq 0$ . Clearly  $p_{\Lambda_{\varepsilon_m}, k}(x)$  converges to  $p_{\Lambda_0, k}(x)$  for  $m \rightarrow \infty$  and  $k = 0, \dots, n$ , and  $\beta_k(\varepsilon_m)$  converges to  $\beta_k(0)$  (cf. formula (13) and Theorem 13). By Theorem 13  $\beta_k(0)$  is positive and the formula  $e^{\lambda_0(t_k(\varepsilon)-a)} \alpha_k(\varepsilon) = \beta_k(\varepsilon)$  now shows that  $\alpha_k(\varepsilon_m)$  converges to the *positive* number  $\alpha_k(0)$ . We define now the Bernstein operator  $B_{\Lambda_0} f$  by (39) with  $\alpha_k := \alpha_k(0)$  for  $k = 0, \dots, n$ . It is easy to see that

$$B_{\Lambda_0}(e^{\lambda_0 x}) = \lim_{\varepsilon \rightarrow 0} B_{\Lambda_\varepsilon}(e^{\lambda_0 x}) = e^{\lambda_0 x}.$$

Clearly

$$f_m(x) := \frac{e^{(\lambda_0 + \varepsilon_m)x} - e^{\lambda_0 x}}{\varepsilon_m} \rightarrow x e^{\lambda_0 x} \text{ for } m \rightarrow \infty$$

and since  $B_{\Lambda_{\varepsilon_m}} f_m(x) = f_m(x)$ , a limit argument shows that  $B_{\Lambda_0}(x e^{\lambda_0 x}) = x e^{\lambda_0 x}$ . The uniqueness is proven in a similar way as in the last proof.  $\square$

## 5. CONVERGENCE OF THE BERNSTEIN OPERATOR

Next we present a sufficient condition for the Bernstein operator  $B_{(\lambda_0, \dots, \lambda_n)}$  to converge to the identity.

**Definition 21.** *For each  $n \in \mathbb{N}$ , let  $\{a(n, k) : k = 0, \dots, n\}$  be a triangular array of complex numbers. We say that  $a(n, k)$  converges uniformly to  $c$  if for each  $\varepsilon > 0$  there exists a natural number  $n_0$  such that  $|a(n, k) - c| < \varepsilon$ , for all  $n \geq n_0$  and all  $k = 0, \dots, n$ .*

The following lemma is implicitly contained in [20, p. 47]. For completeness we include the proof.

**Lemma 22.** *Let  $\gamma > 0$ . For each  $n \in \mathbb{N}$  and each  $j = 0, \dots, n$ , let  $a(n, j) \in (0, 1)$  and  $b(n, j) \in \mathbb{R}$ . Suppose that*

$$(40) \quad \lim_{n \rightarrow \infty} \frac{\log b(n, j)}{\log a(n, j)} = \gamma > 0,$$

*and assume that the convergence is uniform in  $j$ . Define  $A_k(n) = \prod_{j=k}^n a(n, j)$  and  $B_k(n) = \prod_{j=k}^n b(n, j)$ . Then  $\lim_{n \rightarrow \infty} (A_k(n)^\gamma - B_k(n)) = 0$  uniformly in  $k$ .*

*Proof.* We have to show that for each  $\varepsilon_1 > 0$  there exists an  $n_0$  such that for all  $n \geq n_0$  and all  $k = 0, \dots, n$ ,

$$|A_k(n)^\gamma - B_k(n)| < \varepsilon_1.$$

Fix  $\varepsilon_1 > 0$ , and select  $\varepsilon \in (0, 1)$  such that  $1 - \varepsilon^\varepsilon < \varepsilon_1$ ,  $\varepsilon + \varepsilon^\gamma < \varepsilon_1$ ,  $\varepsilon < \gamma$ , and  $\varepsilon^{\gamma-\varepsilon} < \varepsilon_1$ . By (40), there exists an  $n_0$  such that if  $n \geq n_0$  and  $j = 0, \dots, n$ ,

$$\left| \frac{\log b(n, j)}{\log a(n, j)} - \gamma \right| < \varepsilon.$$

Then  $\gamma - \varepsilon < \frac{\log b(n, j)}{\log a(n, j)} < \gamma + \varepsilon$ . Observe that  $\log b(n, j) < 0$  since  $\log a(n, j) < 0$ . So  $(\gamma - \varepsilon) \log a(n, j) > \log b(n, j)$  and  $(\gamma + \varepsilon) \log a(n, j) < \log b(n, j)$ . Hence  $a(n, j)^{\gamma+\varepsilon} \leq b(n, j)$  and  $b(n, j) \leq a(n, j)^{\gamma-\varepsilon}$ . Thus we have proven that

$$(41) \quad A_k(n)^{\gamma+\varepsilon} \leq B_k(n) \text{ and } B_k(n) \leq A_k(n)^{\gamma-\varepsilon}.$$

Next we consider two cases: First assume that  $A_k(n)^\gamma \geq B_k(n)$ . Then, using (41),

$$0 \leq A_k(n)^\gamma - B_k(n) \leq A_k(n)^\gamma - A_k(n)^{\gamma+\varepsilon} = A_k(n)^\gamma (1 - A_k(n)^\varepsilon).$$

If  $A_k(n) \geq \varepsilon$ , then for all  $n \geq n_0$  and all  $k$  we have, using that  $A_k(n) < 1$ ,

$$0 \leq A_k(n)^\gamma - B_k(n) \leq 1 - A_k(n)^\varepsilon \leq 1 - \varepsilon^\varepsilon < \varepsilon_1.$$

If  $A_k(n) < \varepsilon$  for some  $k, n$ , then  $B_k(n) \leq A_k(n)^\gamma \leq \varepsilon^\gamma$  and  $0 \leq A_k(n)^\gamma - B_k(n) \leq \varepsilon + \varepsilon^\gamma < \varepsilon_1$ . In the second case we have  $A_k(n)^\gamma \leq B_k(n)$ . Then, from (41) we get

$$0 \leq B_k(n) - A_k(n)^\gamma \leq A_k(n)^{\gamma-\varepsilon} - A_k(n)^\gamma = A_k(n)^{\gamma-\varepsilon} (1 - A_k(n)^\varepsilon).$$

If  $A_k(n) \geq \varepsilon$  we obtain

$$0 \leq B_k(n) - A_k(n)^\gamma \leq 1 - A_k(n)^\varepsilon \leq 1 - \varepsilon^\varepsilon < \varepsilon_1.$$

Finally, if  $A_k(n) < \varepsilon$  for some  $k, n$ , then  $0 \leq B_k(n) - A_k(n)^\gamma \leq A_k(n)^{\gamma-\varepsilon} - A_k(n)^\gamma \leq \varepsilon^{\gamma-\varepsilon} < \varepsilon_1$ .  $\square$

Next we present our second main result:



**Theorem 23.** *Let  $\lambda_0, \lambda_1, \lambda_2$  be distinct real numbers and let  $\Lambda_n = (\lambda_0, \lambda_1, \dots, \lambda_n)$ , where for  $j = 3, \dots, n$  the complex numbers  $\lambda_j$  are allowed to vary. Suppose each  $E_{(\lambda_0, \dots, \lambda_n)}$  is closed under complex conjugation, and furthermore there exists a positive number  $M$  such that for every  $n \geq 2$  and every  $j = 0, \dots, n$ , we have  $|\operatorname{Im} \lambda_j| \leq M$ . For each  $k \leq n$  set*

$$(42) \quad a(n, k) := \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_2, \dots, \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, \dots, \lambda_n), k}(x)}, \text{ and}$$

$$(43) \quad b(n, k) := \lim_{x \rightarrow b} \frac{p_{(\lambda_0, \lambda_1, \lambda_3, \dots, \lambda_n), k}(x)}{p_{(\lambda_1, \lambda_2, \dots, \lambda_n), k}(x)}.$$

Let  $t_k, k = 0, \dots, n$ , be the uniquely determined points given by Theorem 19. Assume that

$$(44) \quad \lim_{n \rightarrow \infty} t_k - t_{k-1} = 0$$

uniformly in  $k$ , and likewise, that

$$(45) \quad \lim_{n \rightarrow \infty} \frac{\log b(n, k)}{t_k - t_{k+1}} = \lambda_2 - \lambda_0$$

uniformly in  $k$ . Then the Bernstein operator  $B_{(\lambda_0, \dots, \lambda_n)}$  defined in Theorem 19, converges to the identity operator on  $C([a, b], \mathbb{C})$  with the uniform norm.

*Proof.* 1. We remind the reader that  $\tilde{d}_k$  is given by (32), and  $D_k$  by (34). Recall that  $B_{(\lambda_0, \dots, \lambda_n)} f(x) = \sum_{k=0}^n f(t_k) \alpha_k p_{(\lambda_0, \dots, \lambda_n), k}(x)$ , where  $\alpha_k = e^{-\lambda_0(t_k - a)} (-1)^k \tilde{d}_0 \cdots \tilde{d}_{k-1}$ . By construction we have  $B_{(\lambda_0, \dots, \lambda_n)} e^{\lambda_j x} = e^{\lambda_j x}$  for  $j = 0, 1$ . If we show that  $B_{(\lambda_0, \dots, \lambda_n)} e^{\lambda_2(x-a)}$  converges to  $e^{\lambda_2(x-a)}$ , then it follows from the generalized Korovkin theorem for Chebyshev systems that  $B_{\Lambda_n}$  converges to the identity operator (cf. [14], Theorem 8).

2. We may assume that  $\lambda_2 > 0$ . Indeed, we can always translate the vector  $\Lambda_n = (\lambda_0, \dots, \lambda_n)$  by a positive constant  $c$  so that  $\lambda_2 + c > 0$ . Then  $E_{(\lambda_0 + c, \dots, \lambda_n + c)}$  is again closed under complex conjugation; by Proposition 10, the corresponding numbers in (42) and (43) are the same for the translated vector  $c + \Lambda_n$ . If the Bernstein operator converges for  $c + \Lambda_n$ , then so does  $B_{(\lambda_0, \dots, \lambda_n)}$ . Thus, we may assume that  $\lambda_2 > 0$  to begin with.

3. Write (by Theorem 13)  $e^{(x-a)\lambda_2} = \sum_{k=0}^n q_k p_{(\lambda_0, \dots, \lambda_n), k}(x)$ , where  $q_k = (-1)^k \tilde{d}_0 \cdots \tilde{d}_{k-1}$  and

$$(46) \quad \tilde{d}_{n,k} := \lim_{x \rightarrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), k}(x)}{p_{(\lambda_0, \lambda_1, \lambda_3, \dots, \lambda_n), k}(x)}.$$

Thus, for  $\varphi(x) := e^{\lambda_2(x-b)} = e^{\lambda_2(x-a)} e^{\lambda_2(a-b)}$ , by (33) we have

$$(47) \quad B_{(\lambda_0, \dots, \lambda_n)} \varphi(x) - \varphi(x) = \sum_{k=0}^n \left( e^{\lambda_2(t_k - b)} - \frac{q_k}{\alpha_k} e^{\lambda_2(a-b)} \right) \alpha_k p_{(\lambda_0, \dots, \lambda_n), k}(x).$$

Also,  $e^{\lambda_2(t_k - b)} = \prod_{j=k}^{n-1} e^{\lambda_2(t_j - t_{j+1})}$  for  $k \leq n-1$ . Define

$$(48) \quad \tilde{a}(n, j) := e^{t_j - t_{j+1}} < 1.$$

Since  $\varphi(x) = e^{\lambda_2(x-a)}e^{\lambda_2(a-b)}$ , we have

$$(49) \quad \frac{q_k}{\alpha_k} e^{\lambda_2(a-b)} = \frac{\tilde{d}_0 \dots \tilde{d}_{k-1}}{\tilde{d}_0 \dots \tilde{d}_{k-1}} e^{\lambda_0(t_k-a)} e^{\lambda_2(a-b)}.$$

By (17) (applied to  $\lambda_0$  instead of  $\lambda_n$ ) we obtain

$$(50) \quad \frac{q_k}{\alpha_k} e^{\lambda_2(a-b)} = \frac{\tilde{d}_k \dots \tilde{d}_{n-1}}{\tilde{d}_k \dots \tilde{d}_{n-1}} e^{\lambda_0(t_k-b)} = \prod_{j=k}^{n-1} \left( \frac{\tilde{d}_j}{\tilde{d}_j} e^{\lambda_0(t_j-t_{j+1})} \right).$$

Since the left hand side of (50) is real for each  $k = 0, \dots, n-1$ , it is clear that

$$(51) \quad \tilde{b}(n, j) := \frac{\tilde{d}_j}{\tilde{d}_j} e^{\lambda_0(t_j-t_{j+1})} = b(n, j) e^{\lambda_0(t_j-t_{j+1})}$$

is real for all  $j = 0, \dots, n-1$  and all  $n$ . Suppose we know that

$$(52) \quad \frac{\log \tilde{b}(n, j)}{\log \tilde{a}(n, j)} \rightarrow \lambda_2.$$

Then we may use Lemma 22: Set  $\tilde{A}_k(n) = \prod_{j=k}^{n-1} \tilde{a}(n, j)$  and  $\tilde{B}_k(n) = \prod_{j=k}^{n-1} \tilde{b}(n, j)$ . By uniform convergence, given  $\varepsilon > 0$  there exists  $n_0$  such that  $|\tilde{A}_k(n)^{\lambda_2} - \tilde{B}_k(n)| < \varepsilon$  for all  $n \geq n_0$  and all  $k \leq n$ . Note that by (51) and (50),

$$(53) \quad \tilde{B}_k(n) = \frac{q_k}{\alpha_k} e^{\lambda_2(a-b)}.$$

Since  $\tilde{A}_k(n)^{\lambda_2} = e^{\lambda_2(t_k-b)}$  and  $1 \leq e^{\lambda_2(t_k-a)}$ , from (47) we get

$$(54) \quad \begin{aligned} |B_{(\lambda_0, \dots, \lambda_n)} \varphi(x) - \varphi(x)| &\leq \varepsilon \sum_{k=0}^n \alpha_k p_{(\lambda_0, \dots, \lambda_n), k}(x) \\ &\leq \varepsilon \sum_{k=0}^n e^{\lambda_2(t_k-a)} \alpha_k p_{(\lambda_0, \dots, \lambda_n), k}(x) = \varepsilon e^{\lambda_2(b-a)} B_{(\lambda_0, \dots, \lambda_n)} \varphi(x) \end{aligned}$$

for all  $n \geq n_0$  and all  $k \leq n$ . So for every  $x \in [a, b]$ ,

$$\frac{1}{1 + \varepsilon e^{\lambda_2(b-a)}} \varphi(x) \leq B_{(\lambda_0, \dots, \lambda_n)} \varphi(x) \leq \frac{1}{1 - \varepsilon e^{\lambda_2(b-a)}} \varphi(x),$$

proving uniform convergence of  $B_{(\lambda_0, \dots, \lambda_n)} \varphi$  to  $\varphi$  on  $[a, b]$ .

We show next that (52) holds. From formula (35) we obtain

$$(55) \quad a(n, k) = \frac{\tilde{d}_k}{D_k} = e^{(\lambda_0 - \lambda_1)(t_{k+1} - t_k)}.$$

By assumption (44),  $\lim_{n \rightarrow \infty} a(n, k) = 1$  uniformly in  $k$ . Since  $\lambda_0 \neq \lambda_1$ , this implies that

$$(56) \quad \lim_{n \rightarrow \infty} \frac{1 - b(n, k)}{1 - a(n, k)} = \frac{\lambda_2 - \lambda_0}{\lambda_1 - \lambda_0}$$

uniformly in  $k$ . From (48) we have  $\log \tilde{a}(n, k) = t_k - t_{k+1}$ , so by (51),

$$\log \tilde{b}(n, k) = \log (b(n, k) e^{\lambda_0(t_k - t_{k+1})}) = \log b(n, k) + \lambda_0(t_k - t_{k+1}).$$

Now by assumption (45),

$$\frac{\log \tilde{b}(n, k)}{\log \tilde{a}(n, k)} = \frac{\log b(n, k) + \lambda_0(t_k - t_{k+1})}{t_k - t_{k+1}} = \frac{\log b(n, k)}{t_k - t_{k+1}} + \lambda_0 \rightarrow \lambda_2,$$

finishing the proof.  $\square$

## 6. EQUIDISTANT EIGENVALUES

In this section we want to illustrate our results when the eigenvalues in  $(\lambda_0, \dots, \lambda_n)$  are equidistant, i.e., when  $\lambda_j = \lambda_0 + j\omega$  for  $j = 0, \dots, n$ . In this case the elements of  $E_{(\lambda_0, \dots, \lambda_n)}$  are also called  $\mathcal{D}$ -polynomials, see [27] or [11, Remark 2.1]. An important particular instance of  $\mathcal{D}$ -polynomials is the class of scaled trigonometric polynomials, defined for even  $n$  by

$$\text{span } \{1, \sin(2x/n), \cos(2x/n), \sin(4x/n), \cos(4x/n), \dots, \sin x, \cos x\},$$

and for odd  $n$  by

$$\text{span } \{\sin(x/n), \cos(x/n), \sin(3x/n), \cos(3x/n), \dots, \sin x, \cos x\},$$

see [27].

We shall assume that  $\omega \neq 0$  since  $\omega = 0$  leads to the polynomial case, covered by the classical Bernstein theorem. The following result was proved in [27]:

**Proposition 24.** *Let  $\lambda_j = \lambda_0 + j\omega$  for  $j = 0, \dots, n$ , where  $\omega \neq 0$ . Define  $t_k := a + \frac{k}{n}(b - a)$  and  $p_{(\lambda_0, \dots, \lambda_n), k}$  as in (20). Then the operator defined for  $f \in C[a, b]$  by*

$$B_{(\lambda_0, \dots, \lambda_n)} f(x) = \sum_{k=0}^n f(t_k) \frac{n!}{(n-k)!} \frac{\omega^k e^{-\lambda_0(\frac{k}{n}(b-a))}}{(e^{\omega(b-a)} - 1)^k} p_{(\lambda_0, \dots, \lambda_n), k}(x)$$

*satisfies  $B_{(\lambda_0, \dots, \lambda_n)}(e^{\lambda_0 x}) = e^{\lambda_0 x}$  and  $B_{(\lambda_0, \dots, \lambda_n)}(e^{\lambda_n x}) = e^{\lambda_n x}$ .*

*Proof.* Straightforward calculations show that the constants  $d_k$  and  $D_k$  in Theorem 19 are

$$d_k := \lim_{x \rightarrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), k}(x)}{p_{(\lambda_1, \dots, \lambda_n), k}(x)} = -\frac{(n-k)\omega}{1 - e^{\omega(a-b)}} e^{(b-a)(\lambda_0 - \lambda_1)}$$

$$D_k := \lim_{x \rightarrow b} \frac{\frac{d}{dx} p_{(\lambda_0, \dots, \lambda_n), k}(x)}{p_{(\lambda_0, \dots, \lambda_{n-1}), k}(x)} = -\frac{(n-k)\omega}{1 - e^{\omega(a-b)}}.$$

By (35),  $t_k - t_{k-1}$  is defined through

$$(57) \quad e^{(\lambda_0 - \lambda_n)(t_k - t_{k-1})} = \frac{d_{k-1}}{D_{k-1}} = e^{(b-a)(\lambda_0 - \lambda_1)},$$

so  $t_k - t_{k-1} = \frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_n} = \frac{1}{n}(b-a)$ . It follows that  $t_k = a + \frac{k}{n}(b-a)$ . According to (26) we have

$$\alpha_k = e^{-\lambda_0(t_k - a)} (-1)^k d_0 \dots d_{k-1} = e^{-\lambda_0(\frac{k}{n}(b-a))} \frac{e^{(a-b)k\omega} \omega^k}{(1 - e^{\omega(a-b)})^k} \frac{n!}{(n-k)!}.$$

□

The following theorem was proved by S. Morigi and M. Neamtu in [27, p. 137].

**Theorem 25.** *Let  $\mu_0 \neq \mu_1$  be either real numbers or complex conjugates, and in the latter case assume that  $b - a < \pi / |\operatorname{Im} \mu_0|$ . Set  $\Delta := \mu_1 - \mu_0$ , and define  $\lambda_j = \mu_0 + j \frac{1}{2n} \Delta$  for  $j = 0, \dots, 2n$ . Then  $B_{(\lambda_0, \dots, \lambda_{2n})} f(x)$  converges uniformly to  $f$  for all  $f \in C[a, b]$ .*

It is possible to derive Theorem 25 from Theorem 23 (for vectors  $\Lambda_{2n}$  with even index, which guarantees that  $\lambda_n$  is a component of  $\Lambda_{2n}$  for every  $n$ ) applied to the triple  $\lambda_0, \lambda_{2n}$  and  $\lambda_n$  (instead of  $\lambda_0, \lambda_1, \lambda_2$ ). Since the proof is rather technical it is omitted.

## 7. FINAL REMARKS

For  $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots < \lambda_n$ , the Müntz polynomials are defined as elements in the linear space  $V_n$  generated by  $1, x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_n}$ , considered as functions on the interval  $[a, b]$ , where  $a \geq 0$ . Assume  $a > 0$ . Using the transformation  $x = e^t$  we see that  $V_n$  is isomorphic to the linear space  $U_n$  generated by  $1, e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}$ . Clearly  $f \in U_n$  is an exponential polynomial on the interval  $[\ln a, \ln b]$ . To each  $\Lambda_n := (\lambda_0, \dots, \lambda_n)$  we can associate the Bernstein operator  $B_{(\lambda_0, \dots, \lambda_n)}$  for the interval  $[\ln a, \ln b]$ . Convergence of  $B_n$  to the identity operator implies that the union of the spaces  $U_n, n \in \mathbb{N}$ , is dense in  $C[\ln a, \ln b]$ . It is well known (see [2, p. 180]) that this entails

$$(58) \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

In particular, it follows that Theorem 23 does not extend to the case of arbitrary vectors  $\Lambda_n = (\lambda_0, \dots, \lambda_n)$ . It would be interesting to derive from Theorem 23 a Bernstein type result for Müntz polynomials over  $[\ln a, \ln b]$ ,  $a > 0$ . Let us mention that the results in

[12] for Müntz polynomials over  $[0, 1]$  are of a different type, since the basic functions used there do not form a Bernstein basis in our sense.

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J.M. ALDAZ: DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, EDIFICIO VIVES, LUIS DE ULLOA S/N., 26004 LOGROÑO, SPAIN.

*E-mail address:* aldaz@dmcunirioja.es

O. KOUNCHEV: INSTITUTE OF MATHEMATICS AND INFORMATICS, BULGARIAN ACADEMY OF SCIENCES, 8 ACAD. G. BONCHEV STR., 1113 SOFIA, BULGARIA.

*E-mail address:* kounchev@gmx.de

H. RENDER: DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, EDIFICIO VIVES, LUIS DE ULLOA S/N., 26004 LOGROÑO, ESPAÑA.

*E-mail address:* render@gmx.de